

On the k -power propagation time of a graph

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Abstract

In this paper, we characterize all graphs G with extreme k -power propagation time $|G| - 1$ or $|G| - 2$ for $k \geq 1$, and $|G| - 3$ for $k \geq 2$. We determine all trees T whose 1-power propagation time (also called power propagation time or *standard* power propagation time) is $|T| - 3$. Partial characterizations of graphs with k -power propagation time equal to 1 are also established. Finally, we consider the effects of edge subdivisions and edge contractions on the standard power propagation time of a graph, and give an upper bound on the sum of the standard power propagation time of a graph and its complement.

1 Introduction

Phasor Measurement Units (PMUs) are machines used by energy companies to monitor the electric power grid system. They are placed at selected electrical nodes (locations at which transmission lines, loads, and generators are connected) within the system. Due to the high cost of the machines, an extensive amount of research has been devoted to minimizing the placement of the PMUs while maintaining the ability to observed the entire system. In [8], Haynes et al. studied this problems in terms of graphs.

An electric power grid system is modeled by a graph by letting vertices represent the electrical nodes (also called buses), and edges represent transmissions lines between nodes. The *power domination process* is defined as follows [8]: A PMU placed at a vertex measures the voltage and phasor angle at that vertex, as well as the incident edges and the vertices at the endpoints of these edges. These vertices and edges are said to be *observed*. [8] The rest of the system is observed according to the following propagation rules:

1. Any vertex that is incident to an observed edge is observed.
2. Any edge joining two observed vertices is observed.

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3. If a vertex is incident to a total of $t > 1$ edges and if $t - 1$ of these edges are observed, then all t of these edges are observed.

We state an equivalent formulation of the power domination process as done in [7]. Let $G = (V, E)$ be a graph and $v \in V(G)$. (All graphs discussed are simple graphs.) The *open neighborhood* of v , denoted $N(v)$, is given by $N(v) = \{u \in V(G) | vu \in E(G)\}$. The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, $N(S) = \cup_{s \in S} N(s)$ and $N[S] = \cup_{s \in S} N[s]$.

For a set $S \subseteq V(G)$, define the following sets:

1. $S^{[1]} = N[S]$.
2. For $t \geq 1$, $S^{[t+1]} = S^{[t]} \cup \{w \in V(G) | \exists v \in S^{[t]}, w \in N(v) \cap (V(G) \setminus S^{[t]}) \text{ and } |N(v) \cap (V(G) \setminus S^{[t]})| = 1\}$.

For vertices w and v given in 2. above, we say v *forces* w . Computing $S^{[1]}$ is the *domination step* and the computations of $S^{[t+1]}$ (for $t \geq 1$) are the *propagation steps*. A set S is said to be a *power dominating set* if there exists an l such that $S^{[l]} = V(G)$. The *power domination number* of G , denoted $\gamma_P(G)$, is the minimum cardinality over all power dominating sets of G . Power domination was first introduced and studied in [8].

A set $S \subseteq V(G)$ is a *dominating set* if for each $v \in V(G) \setminus S$, there exists a $u \in S$ such that v is adjacent to u . The *domination number* of a graph G , denoted $\gamma(G)$, is the minimum cardinality over all dominating sets of G . Note that each dominating set is a power dominating set, so $\gamma_P(G) \leq \gamma(G)$ [8].

The authors of [4] introduced the following generalization of power domination, known as *k-power domination*. Let $k \geq 1$. For a set $S \subseteq V(G)$, define the following sets:

1. $S^{[1]} = N[S]$.
2. For $t \geq 1$, $S^{[t+1]} = S^{[t]} \cup \{w \in V(G) | \exists v \in S^{[t]}, w \in N(v) \cap (V(G) \setminus S^{[t]}) \text{ and } |N(v) \cap (V(G) \setminus S^{[t]})| \leq k\}$.

A set S is said to be a *k-power dominating set* if there exists an l such that $S^{[l]} = V(G)$. Note that the case with $k = 1$, the set is simply a power dominating set. The *k-power domination number* of G , denoted $\gamma_{P,k}(G)$, is defined to be the minimum cardinality over all *k-power dominating sets* of G , and $\gamma_{P,k}(G) \leq \gamma_P(G) \leq \gamma(G)$ for all $k \geq 1$ [4].

Let S be a *k-power dominating set*. The *k-power propagation time* of G with S , denoted $\text{ppt}_k(G, S)$ (or $\text{ppt}(G, S)$ when $k = 1$), is the smallest l such that $S^{[l]} = V(G)$. In the case with $k = 1$, we simply write *power propagation time*. The *minimum k-power propagation time* of G , denoted $\text{ppt}_k(G)$ (or $\text{ppt}(G)$ when $k = 1$), is given by

$$\text{ppt}_k(G) = \min\{\text{ppt}_k(G, S) | S \text{ is a minimum } k\text{-power dominating set}\}.$$

The *maximum k-power propagation time* of G , denoted $\text{PPT}_k(G)$, is given by

$$\text{PPT}_k(G) = \max\{\text{ppt}_k(G, S) | S \text{ is a minimum } k\text{-power dominating set}\}.$$

The k -power propagation time interval of G is defined as

$$[\text{ppt}_k(G), \text{PPT}_k(G)] = \{\text{ppt}_k(G), \text{ppt}_k(G) + 1, \dots, \text{PPT}_k(G)\}.$$

If there exists a minimum k -power dominating set S such that $\text{ppt}_k(G, S) = r$ for each r in the k -power propagation time interval, we say that the interval is *full*. A natural question is whether or not the k -propagation time interval is full for all graphs. The k -propagation time interval need not be full, as demonstrated by example 4.5 in [7]

A minimum k -power dominating set S of a graph G is *efficient* if $\text{ppt}_k(G, S) = \text{ppt}_k(G)$.

Zero forcing is a game played on a graph using the following *color change rule*: Let B be a set of vertices of G that are colored blue with $V - B$ colored white. If v is a blue vertex and u is the only neighbor of v that is colored white, then change the color of u to blue. For a set B of vertices that are initially colored blue, the set of blue vertices that results from applying the color change rule until no more color changes are possible is the *final coloring of B* . A set B is said to be a *zero forcing set* if the final coloring of B is the entire vertex set $V(G)$. The minimum cardinality over all zero forcing sets of G is the *zero forcing number of G* , denoted $Z(G)$. The zero forcing number was first introduced and studied in [1] as an upper bound on the linear algebraic parameter of a graph known as the maximum nullity, and independently in [3] to study the control of quantum systems.

For a given zero forcing set B of G , construct the final coloring. The set of forces that are performed is called a *set of forces of B* . Given a set of forces \mathcal{F} , a *forcing chain of \mathcal{F}* is a sequence of vertices (v_1, \dots, v_k) such that for $i = 1, \dots, k-1$, v_i forces v_{i+1} in \mathcal{F} ($k = 1$ is permitted) [9]. A *maximal forcing chain* is a forcing chain that is not a proper subsequence of another forcing chain [9]. Note that maximal forcing chains correspond to induced paths in the graph G .

Observation 1. [2] A set S is a power dominating set of G if and only if $N[S]$ is a zero forcing set of G , and $N(S) \setminus S$ is a zero forcing set for $G - S$.

The authors of [9] introduced the *propagation time* of a zero forcing set and of a graph (given below). Many of the results given in this paper were motivated by the study of the propagation time of a graph.

Definition 2. [9] Let $G = (V, E)$ be a graph and B a zero forcing set of G . Define $B^{(0)} = B$, and for $t \geq 0$, $B^{(t+1)}$ is the set of vertices w for which there exists a vertex $b \in \cup_{s=0}^t B^{(s)}$ such that w is the only neighbor of b not in $\cup_{s=0}^t B^{(s)}$. The *propagation time of B in G* , denoted $\text{pt}(G, B)$, is the smallest integer t_0 such that $V = \cup_{t=0}^{t_0} B^{(t)}$. The *minimum propagation time of G* is $\text{pt}(G) = \min\{\text{pt}(G, B) | B \text{ is a minimum zero forcing set of } G\}$, and the *maximum propagation time of G* is $\text{PT}(G) = \max\{\text{pt}(G, B) | B \text{ is a minimum zero forcing set of } G\}$.

Note that this definition is analogous to the definition of the power propagation time of a graph.

We use P_n, C_n , and K_n to denote the path on n vertices, the cycle on n vertices, and the complete graph on n vertices, respectively. The notation $K_n - e$ represents the complete graph on n vertices minus an edge, and $L(s, t)$ denotes the lollipop graph consisting of a complete graph on s vertices and a path on t vertices connected with a bridge. The graph $K_{s,t}$ is the complete bipartite graph with bipartition X, Y where $|X| = s$ and $|Y| = t$.

Let $G = (V, E)$ be a graph and $e = uv \in E(G)$. The graph resulting from *subdividing* the edge $e = uv$, denoted G_e , is obtained from G by adding a new vertex w such that $V(G_e) = V(G) \cup \{w\}$ and $E(G_e) = (E(G) \setminus \{uv\}) \cup \{uw, vw\}$. To *contract* the edge $e = uv$ is to identify vertices u and v as a single vertex w such that $N(w) = (N(u) \cup N(v)) \setminus \{u, v\}$. The graph obtained from G by contracting the edge e is denoted by G/e .

A *spider* or *generalized star* is a tree formed from a $K_{1,n}$ by subdividing any number of its edges any number of times. We use $sp(i_1, i_2, \dots, i_n)$ to denote the spider obtained from $K_{1,n}$ by subdividing edge e_j a total of $i_j - 1$ times for $1 \leq j \leq n$.

Observation 3. Let G be a graph and S a k -power dominating set of G . Then,

$$\text{ppt}_k(G, S) \leq |G| - |S| \quad (1)$$

and

$$\text{ppt}_k(G, S) - 1 \leq |G| - |N[S]| \quad (2)$$

because at least 1 vertex must be forced at each step.

In Section 3, we characterize all graphs on n vertices whose k -power propagation time is $n - 1$ and $n - 2$ for $k \geq 1$. For $k \geq 2$, we characterize all graphs whose k -power propagation time is $n - 3$, and for $k = 1$ we give partial characterizations for such graphs. In Section 4, we give a characterization of graphs with k -power propagation time 1 for $k \geq 1$. An upper bound on the power domination number of a graph and its complement is given in Section 5, and in Section 6 we consider the effects of edge subdivision and edge contraction on the power propagation time of a graph.

2 Preliminaries

In this section, we give preliminary results that will be used as tools for characterizing graphs with high and low k -power propagation times.

It is clear that the k -power domination number of the graphs P_n and C_n is 1 for all $k \geq 1$. We determine the k -power propagation time for these graphs.

Proposition 4. Let P_n be the path on n vertices. Then $\text{ppt}_k(P_n) = \lfloor \frac{n}{2} \rfloor$ for all $k \geq 1$.

Proof. Let $G = P_n$. Any one vertex of G is a minimum k -power dominating set. Label the vertices of G with v_1, \dots, v_n where $\{v_i, v_{i+1}\} \in E(G)$ for $i \in \{1, \dots, n-1\}$. For any vertex v_t , $\text{ppt}_k(G, \{v_t\}) = \max\{t-1, n-t\}$. It follows that for n odd, $\text{ppt}_k(G) \geq \frac{n-1}{2}$, and equality is obtained by choosing the k -power dominating set to be $\{v_t\}$ where $t = \frac{n+1}{2}$. For n even $\text{ppt}_k(G) \geq \frac{n}{2}$, and equality is obtained by choosing the k -power dominating set $\{v_t\}$ with $t \in \{\frac{n}{2}, \frac{n+1}{2}\}$. \square

Proposition 5. *Let C_n be the cycle on n vertices. Then $\text{ppt}_k(C_n) = \lfloor \frac{n}{2} \rfloor$ for all $k \geq 1$.*

Proof. Let $G = C_n$. Any one vertex of G is an efficient k -power dominating set. For n even, $\text{ppt}_k(G, \{v\}) = \frac{n}{2}$ for all $v \in V(G)$. For n odd, $\text{ppt}_k(G, \{v\}) = \frac{n-1}{2}$ for all $v \in V(G)$. \square

Remark 6. It is a well known fact that for a connected graph G of order at least 3, there exists an efficient k -power dominating set of G in which every vertex has degree at least 2. For if v is a leaf of an efficient k -power dominating set S and $vw \in E(G)$, then w is not a leaf since G is connected and $G \neq K_2$, $S' = (S \setminus \{v\}) \cup \{w\}$ is a minimum k -power dominating set, and $\text{ppt}_k(G, S') \leq \text{ppt}_k(G, S)$. Repeating this process for each leaf in S , we obtain an efficient k -power dominating set of G with no leaves.

Lemma 7. [4] *Let $k \geq 1$ and let G be a connected graph with $\Delta(G) \geq k+2$. Then there exist a minimum k -power dominating set S of G such that $\deg(s) \geq k+2$ for each $s \in S$.*

Note that $\Delta(G) \geq k+2$ does not guarantee that there exists an efficient k -power dominating set S such that $\deg(s) \geq k+2$ for each $s \in S$. This is demonstrated in the following example with $k = 1$.

Example 8. Let G be the graph on $n+2$ vertices ($n \geq 5$) obtained from a path (v_1, v_2, \dots, v_n) by adding a leaf to v_2 and adding a leaf to v_3 . Then $S = \{v_2, v_3\}$ is the unique power dominating set such that $\deg(s) \geq 3$ for each $s \in S$, but for $S' = \{v_2, v_4\}$, $n-6 = \text{ppt}(G, S') < \text{ppt}(G, S) = n-5$.

Remark 9. For any $3 \leq t \leq k+2$, if G is connected with $\Delta(G) \geq t$, then there exists a minimum k -power dominating set S such that every vertex in S has degree at least t .

3 High k -power propagation time

In [9], it is shown that $\text{pt}(G) = |G| - 1$ if and only if G is a path. The authors also characterize all graphs G with $\text{pt}(G) = |G| - 2$. Here we consider graphs with high k -power propagation times. We first characterize all graphs G with $\text{ppt}_k(G) = |G| - 1$ or $\text{ppt}_k(G) = |G| - 2$.

Theorem 10. For a graph G and $k \geq 1$, $\text{ppt}_k(G) = |G| - 1$ if and only if $G = K_1$ or $G = K_2$.

Proof. Let S be an efficient k -power dominating set of G . Since $\text{ppt}_k(G) = |G| - 1$, then $S = \{s\}$ for some $s \in V(G)$, and G is connected. Note that at most 1 vertex may be forced at each step, including the domination step, so $\deg(s) \leq 1$. By Remark 6, $|G| \leq 2$, so $G = K_1$ or $G = K_2$. \square

Theorem 11. Let $k \geq 1$ and let G be a graph with $\text{ppt}_k(G) = |G| - 2$. Then $G \in \{K_1 \cup K_1, K_1 \cup K_2, P_3, P_4, C_3, C_4\}$.

Proof. Since $\text{ppt}_k(G) = |G| - 2$, then for any minimum k -power dominating set S , $|S| \leq 2$ and $|N[S]| \leq 3$. Suppose $\Delta(G) \geq 3$. If G is connected, by Remark 9, there exists a minimum k -power dominating set S such that each $s \in S$ has degree at least 3, contradicting $|N[S]| \leq 3$. If G is disconnected, then we apply Remark 9 to a connected component of G that contains a vertex of maximum degree to obtain that there exists a minimum k -power dominating set S of G that contains a vertex of degree at least 3. This contradicts $|N[S]| \leq 3$. So $\Delta(G) \leq 2$ and G is the union of cycles and paths. Since $\gamma_{P,k}(G) \leq 2$, then G has at most 2 components. If G has exactly one component, G is a path or a cycle, and it follows from Propositions 4 and 5 that $G \in \{P_3, P_4, C_3, C_4\}$. Suppose G has 2 components. Since $|N[S]| \leq 3$, one component is K_1 , and by Remark 6 (or Theorem 10), the other component is K_1 or K_2 . \square

Next we consider graphs on n vertices whose k -power propagation time is $n - 3$. The case with $k = 1$ behaves differently than the cases with $k \geq 2$, so we first consider the latter.

We use \mathfrak{G} to denote the family of connected graphs G on 5 vertices with $\Delta(G) = 3$.

Theorem 12. Let $k \geq 2$ and let G be a graph with $\text{ppt}_k(G) = |G| - 3$. Then $G \in \{P_5, P_6, C_5, C_6, sp(1, 1, 1), L(3, 1), K_4 - e, K_4, K_1 \cup P_3, K_1 \cup P_4, K_1 \cup C_3, K_1 \cup C_4, K_2 \cup K_2, \overline{K_3}, \overline{K_2} \cup K_2\} \cup \mathfrak{G}$.

Proof. Since $\text{ppt}_k(G) \leq |G| - 3$, then for any minimum k -power dominating set S , $|S| \leq 3$ and $|N[S]| \leq 4$. It follows from Remark 9 that $\Delta(G) \leq 3$.

If $\Delta(G) \leq 2$, then G is the union of paths and cycles. Since $\gamma_{P,k}(G) \leq 3$, G has at most 3 components. If G is connected, it follows from Propositions 4 and 5 that $G \in \{P_5, P_6, C_5, C_6\}$. Suppose G has two connected components, G_1 and G_2 , and suppose $|G_1| \geq 3$. By applying Remark 6 to G_1 , there exists an efficient k -power dominating set S of G such that $|N_{G_1}[S]| \geq 3$, where $N_{G_1}[S] = N[S] \cap V(G_1)$. Since $|N[S]| \leq 4$, we have $G_2 = K_1$, $\text{ppt}_k(G_1) = |G_1| - 2$, and it follows from Theorem 12 that $G \in \{K_1 \cup P_3, K_1 \cup P_4, K_1 \cup C_3, K_1 \cup C_4\}$. If $|G_1| \leq 2$ and $|G_2| \leq 2$, then $G = K_2 \cup K_2$.

If G has 3 connected components, it follows from $|N[S]| \leq 4$ that $G \in$

$\{\overline{K_3}, \overline{K_2} \cup K_2\}$.

Suppose $\Delta(G) = 3$. Let S be a minimum k -power dominating set such that every vertex in S has degree at least 3. Since $|N[S]| \leq 4$, then $S = \{s\}$ for some $s \in V(G)$. Let $N[S] = \{s, s_1, s_2, s_3\}$. If $|G| = 4$, then $G \in \{\text{sp}(1, 1, 1), L(3, 1), K_4 - e, K_4\}$.

Next, we show that if $|G| > 4$, then $|G| = 5$. Since $|N[S]| = 4$ and $\text{ppt}_k(G) = |G| - 4$, then after the domination step, exactly one force is performed during each step. Without loss of generality, suppose s_1 forces v in step 2.

Claim 1: For $i \in \{1, 2, 3\}$, if $u \in N(s_i)$, then $u \in \{s, s_1, s_2, s_3, v\}$. To see this, recall that $\Delta(G) = 3$. So if s_1 has a neighbor u not in $\{s, s_2, s_3, v\}$, it has exactly one such neighbor, and it will force u and v in step 2, contradicting $\text{ppt}_k(G) = |G| - 3$. Similarly, if s_i (for $i = 2, 3$) has a neighbor u not in $\{s, s_1, s_2, s_3, v\}$, it has at most two such neighbors, so s_1 will force v in step 2 and s_i will force u in step 2, contradicting $\text{ppt}_k(G) = |G| - 3$.

Claim 2: Vertex v has no neighbor not in $\{s_1, s_2, s_3\}$. To see this, suppose v has a neighbor u not in $\{s_1, s_2, s_3\}$. Since $\Delta(G) = 3$ and v is adjacent to s_1 by assumption, then v has at most two such neighbors. Then $\{s_1\}$ is a minimum k -power dominating set with $\text{ppt}_k(G, \{s_1\}) \leq |G| - 4$ since s_1 will dominate $\{v, s\}$ in step 1, and if necessary, s will force $\{s_2, s_3\}$ in step 2 and v will force u in step 2.

Therefore, G is a connected graph on 5 vertices with $\Delta(G) = 3$. Also note that all connected graphs on 5 vertices with maximum degree 3 have $\text{ppt}_k(G) = 2$ (for $k \geq 2$). This completes the proof. \square

Next we consider graphs with $\text{ppt}(G) = |G| - 3$. We characterize all trees with $\text{ppt}(G) = |G| - 3$ and all graphs with $\text{ppt}(G) = |G| - 3$ and $\gamma_P(G) \in \{2, 3\}$. In Figure 1, we provide some graphs (including infinite families) with $\text{ppt}(G) = |G| - 3$ and $\gamma_P(G) = 1$, but characterizing all such graphs is less tractable.

Proposition 13. *Let T be a tree with $\text{ppt}(T) = |T| - 3$. Then $T = P_5$, $T = P_6$, or $T = \text{sp}(1, 1, k)$, for some $k \geq 1$.*

Proof. If $\Delta(T) \leq 2$, then T must be a path, and by Proposition 4, $T = P_5$ or $T = P_6$. Suppose $\Delta(T) \geq 3$. From Lemma 7, there exists a minimum power dominating set S such that each vertex in S has degree at least 3. Since each vertex in S has degree at least 3, then $|N[S]| \geq 4$ and $\text{ppt}(T, S) \leq |T| - 3$. This implies that $\text{ppt}(T, S) = |T| - 3$ since $\text{ppt}(T) = |T| - 3$. So, $|S| = 1$ and $|N[S]| = 4$.

Let $S = \{s\}$ and $N[S] = \{s, s_1, s_2, s_3\}$. If $\text{ppt}(T) = 1$, then $|T| = 4$ and $T = \text{star}(1, 1, 1)$. Suppose $\text{ppt}(T) \geq 2$. Since $|N[S]| = 4$, then for steps 2 through $\text{ppt}(T)$, there is exactly one force per step. Consider a set of forces and the corresponding maximal forcing chains $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ for the zero forcing set $\{s_1, s_2, s_3\}$ in $T' = T - s$. Recall that $\mathcal{C}_1, \mathcal{C}_2$, and \mathcal{C}_3 correspond to induced paths in T' . Also note that there is no edge between any two of these chains, as

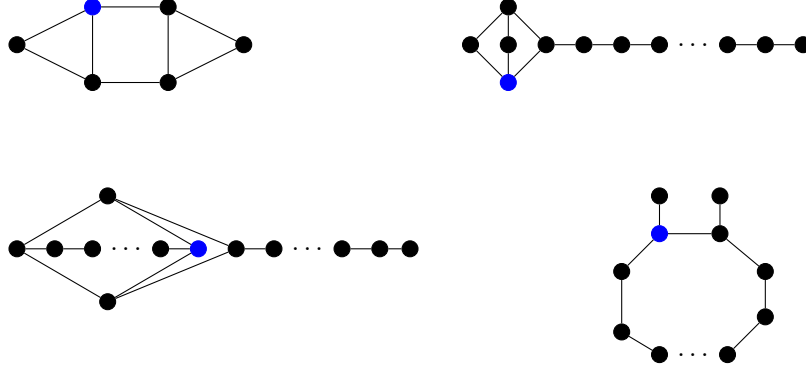


Figure 1: Graphs G with $\text{ppt}(G) = |G| - 3$ and $\gamma_P(G) = 1$. An efficient power dominating set in blue.

this would create a cycle in T . Since there is exactly 1 force per step, it follows that two of the chains must consist of a single vertex. So, $T = \text{star}(1, 1, k)$ for some $k \geq 1$.

□

Theorem 14. *Let G be a graph with $\text{ppt}(G) = |G| - 3$ and $\gamma_p(G) \in \{2, 3\}$. Then $G \in \mathcal{F}$ where $\mathcal{F} = \{\overline{K_3}, \overline{K_2} \cup K_2, K_1 \cup C_3, K_1 \cup P_3, K_1 \cup P_4, K_1 \cup C_4, K_2 \cup K_2\}$.*

Proof. For any minimum power dominating set S of G , $|S| \leq 3$ and $|N[S]| \leq 4$. Suppose G is connected. If $\Delta(G) \geq 3$, by Lemma 7, G has a minimum power dominating set S such that each $s \in S$ has degree at least 3. Since $|S| \in \{2, 3\}$, this gives $|N[S]| \geq 5$. If $\Delta(G) \leq 2$, then G is a path or a cycle with $\gamma_p(G) = 1$. Thus, G has at least two components.

Suppose G has only two components, G_1 and G_2 . Without loss of generality, if G_1 has at least 3 vertices and $\Delta(G_1) \geq 3$, by applying Lemma 7 to G_1 , it follows that there exists a minimum power dominating set S of G with $|N[S]| \geq 5$. So, $\Delta(G_i) \leq 2$, G_i is a path or a cycle, and $\gamma_p(G) = 2$. If G_1 is a path on at least 3 vertices or a cycle, then $G_2 = K_1$ (since $|N[S]| \leq 4$) and $\text{ppt}(G) = \text{ppt}(G_1) = |G_1| - 2$. By Proposition 12, $G_1 \in \{P_3, P_4, C_3, C_4\}$. Otherwise, $G = K_2 \cup K_2$.

If G has three components G_1, G_2, G_3 , then $\gamma_p(G) = 3$ and exactly one force is performed at each step. So, $G_1 = G_2 = K_1$, and $\text{ppt}(G) = \text{ppt}(G_3) = |G_3| - 1$. By Proposition 10, $G_3 \in \{K_1, K_2\}$.

□

4 Low k -power propagation time

In this section we study graphs with low k -propagation time. If G is a graph with k -propagation time 1, then any efficient k -power dominating set of G is also a dominating set, so $\gamma_{P,k}(G) = \gamma(G)$. It was shown in [8] that every graph H is an induced subgraph of a graph G with $\gamma_P(G) = \gamma(G)$. Thus, there is no forbidden induced subgraph characterization of graphs with power propagation time 1.

Let G be a graph. For $k \geq 1$, a vertex v in $V(G)$ is called a *k -strong support vertex* if v is adjacent to $k+1$ or more leaves. A 1-strong support vertex is also known as a *strong support vertex* and was originally defined in [8].

Remark 15. Note that every k -strong support vertex of a graph G is in every minimum dominating set of G . Also, if S is a k -power dominating set of G and v is a k -strong support vertex of G then either v is in S or all but k of the leaves adjacent to v are in S . So $\gamma_{P,k}(G)$ is at least the number of k -strong support vertices in G . Since $\gamma_{P,k}(G) \leq \gamma(G)$, it follows that if S is a dominating set of G such that every vertex in S is a k -strong support vertex, then S is minimum and unique, $\gamma_{P,k}(G) = \gamma(G)$, and $\text{ppt}_k(G) = 1$.

For a minimum k -power dominating set S and a vertex v in S , the *private neighborhood* of v with respect to S , denoted $pn[v, S]$, is the set $N[v] - N[S - \{v\}]$. Every vertex of $pn[v, S]$ is called a *private neighbor* of v with respect to S , and A_v denotes the set $V - (S \cup pn[v, S])$ [8].

For $k \geq 1$, let \mathfrak{F}_k be the set of graphs defined by $\mathfrak{F}_k = \{C_3, K_{2,k+1}, K_2 \vee \overline{K_k}\}$.

For graphs G and H , G is said to be *H -free* if G does not contain H as an induced subgraph. If \mathcal{H} represents a family of graphs, then G is said to be *\mathcal{H} -free* if for all $H \in \mathcal{H}$, G does not contain H as an induced subgraph.

The next theorem and proof is a generalization of Theorem 9 given in [8].

Theorem 16. *For $k \geq 1$, let G be a connected graph on at least $k+2$ vertices that is \mathfrak{F}_k -free. Then $\text{ppt}_k(G) = 1$ if and only if G has a minimum dominating set S such that every vertex in S is a k -strong support vertex.*

Proof. If G has a dominating set S such that each vertex in S is a k -strong support vertex, then by Remark 15, $\gamma_{P,k}(G) = \gamma(G)$ and $\text{ppt}_k(G) = 1$. Conversely, suppose $\text{ppt}_k(G) = 1$ (i.e. $\gamma_{P,k}(G) = \gamma(G)$). To obtain a contradiction, suppose S is a minimum dominating set of G such that there exists a vertex $v \in S$ that is not a k -strong support vertex. If $pn[v, S] = \emptyset$, then $S - \{v\}$ is a smaller dominating set. Suppose that $pn[v, S] = \{v\}$. Then $S - \{v\}$ dominates $V - \{v\}$, and since G is connected, $S - \{v\}$ is a smaller k -power dominating set. So there exists a vertex $w \neq v$ in $pn[v, S]$.

Suppose $pn[v, S]$ contains no leaves. Since G is \mathfrak{F}_k -free, then each vertex in $pn[v, S]$ that is not v is adjacent to a vertex in A_v , and for each $u \in A_v$, $|N(u) \cap (pn[v, S] \cup \{v\})| \leq k$. It follows that $S - \{v\}$ is a k -power dominating set of G since $S - \{v\}$ dominates A_v , each vertex in $pn[v, S] - \{v\}$ is forced by a neighbors in A_v , and if necessary, any vertex in $pn[v, S] - \{v\}$ can force $\{v\}$.

So there must exist at least one leaf in $pn[v, S]$.

Suppose vertices u_1, \dots, u_t ($1 \leq t \leq k-1$) are leaves $pn[v, S]$. If $pn[v, S] = \{u_1, \dots, u_t, v\}$, then v has a neighbor in A_v since G is connected, and we again have that $S - \{v\}$ is a k -power dominating set of G . If there exists a $w \in pn[v, S]$ such that $w \notin \{u_1, \dots, u_t, v\}$, then w has a neighbor in A_v and $S - \{v\}$ is a k -power dominating set of G . \square

For the remainder of the paper, we focus our attention on the standard power propagation time, $ppt(G)$.

5 Nordhaus-Gaddum sum upper bound

In this section we show that for all graphs on n vertices, $ppt(G) + ppt(\overline{G}) \leq n+2$. We also conjecture that n is the least upper bound, and demonstrate an infinite family of graphs with $ppt(G) + ppt(\overline{G}) = n$ for each G in the family.

Proposition 17. *Let G be a graph on n vertices. Then $ppt(G) + ppt(\overline{G}) \leq n+2$.*

Proof. If G has no edges, then $ppt(G) = 0$ and $ppt(\overline{G}) = 1$ so the claim holds. Suppose G has an edge. Let S be an efficient power dominating set of G . Note that $N[S]$ is a zero forcing set of G , but it is not minimum. To see this, consider a fixed $s \in S$ (such that $\deg(s) \geq 1$) and a vertex $v_s \in N(s)$. By removing v_s , $N[S] \setminus \{v_s\}$ is also a zero forcing set, so $Z(G) + 1 \leq |N[S]|$. Similarly, $Z(\overline{G}) + 1 \leq |N[S']|$, where S' is an efficient power dominating set of \overline{G} . It follows from inequality (2) that $ppt(G) + ppt(\overline{G}) \leq 2n - (Z(G) + Z(\overline{G}))$, and since $n - 2 \leq Z(G) + Z(\overline{G})$ from [6], then $ppt(G) + ppt(\overline{G}) \leq |G| + 2$. \square

We have not found a graph G such that $ppt(G) + ppt(\overline{G}) = n+1$ or one such that $ppt(G) + ppt(\overline{G}) = n+2$. We have computationally checked all connected graphs on at most 10 vertices and found several graphs with $ppt(G) + ppt(\overline{G}) = n$. Evidence suggests that this is the least upper bound for all graphs. The next example gives an infinite family of graphs such that $ppt(G) + ppt(\overline{G}) = n$ for all graphs in the family.

Example 18. Let G_9 denote the graph given in the Figure 2. For $n \geq 10$, let G_n be a graph on n vertices constructed from G_{n-1} by adding an n^{th} vertex and adding the edges $\{v_{n-2}, v_n\}$ and $\{v_{n-1}, v_n\}$. Note that the set $V(G_n) \setminus \{v_2, v_3\}$ is not a power dominating set of G_n since $N(v_2) = N(v_3)$. So for every power dominating set S of G_n , $N[S]$ must contain either v_2 or v_3 . Also note that the sets $\{v_2\}$ and $\{v_3\}$ are minimum power dominating sets of G_n with $ppt(G_n, v_2) = ppt(G_n, v_3) = n - 3$. Thus, $\gamma_P(G) = 1$. For $6 \leq i \leq n$, the set $\{v_i\}$ is not a power dominating set since $v_2, v_3 \notin N[\{v_i\}]$. Furthermore, it follows from inspection that the sets $\{v_1\}$, $\{v_4\}$, and $\{v_5\}$ are not power dominating sets. Thus, $ppt(G_n) = n - 3$.

Similarly, in $\overline{G_n}$, note that $V(\overline{G_n}) \setminus \{v_2, v_3\}$ is not a power dominating set

of $\overline{G_n}$ since $N(v_2) \setminus \{v_3\} = N(v_3) \setminus \{v_2\}$. Thus, for each power dominating set S' of $\overline{G_n}$, $N[S']$ must contain v_2 or v_3 . Furthermore, the sets $\{v_2\}$ and $\{v_3\}$ are power dominating sets with $\text{ppt}(\overline{G_n}, \{v_2\}) = \text{ppt}(\overline{G_n}, \{v_3\}) = 4$, so $\gamma_P(G) = 1$. For $i \in \{1, 4, 5\}$, note that the set $\{v_i\}$ is not a power dominating set since $v_2, v_3 \notin N[\{v_i\}]$. It follows from inspection that the set $\{v_i\}$ is not a power dominating set for $7 \leq i \leq n-2$ and the sets $\{v_{n-1}\}$ and $\{v_n\}$ are power dominating sets of $\overline{G_n}$ with $\text{ppt}(\overline{G_n}, \{v_{n-1}\}) = \text{ppt}(\overline{G_n}, \{v_n\}) = 3$. So, $\text{ppt}(G_n) + \text{ppt}(\overline{G_n}) = n$ for all $n \geq 9$.

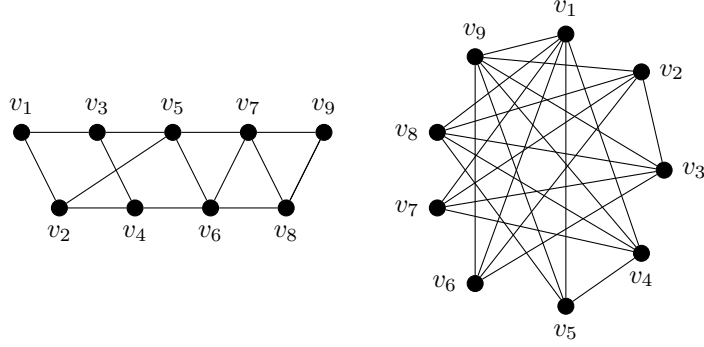


Figure 2: Graphs G_9 (left) and $\overline{G_9}$ (right) in Example 18.

Conjecture 19. For all graphs G on n vertices, $\text{ppt}(G) + \text{ppt}(\overline{G}) \leq n$.

We now show that the conjecture is true for all graphs with at least one leaf.

Proposition 20. Let $G \neq P_4$ be a connected graph on n vertices that has a leaf. Then $\text{ppt}(G) + \text{ppt}(\overline{G}) \leq n - 1$ and this bound is tight. For $G = P_4$, $\text{ppt}(G) + \text{ppt}(\overline{G}) = n = 4$.

Proof. The claim holds when $n \leq 2$, so let $n \geq 3$. We first show that $\text{ppt}(\overline{G}) \leq 2$. Let $uv \in E(G)$ such that v is a leaf. If $\deg(u) = n - 1$, then $\{v, u\}$ is an efficient power dominating set for \overline{G} and $\text{ppt}(\overline{G}) = 1$. If $\deg(u) \neq n - 1$, then $\{v\}$ is an efficient power dominating set for \overline{G} , and $\text{ppt}(\overline{G}) = 2$.

Suppose $\Delta(G) \geq 3$. By Lemma 7, G has a minimum power dominating set S such that each vertex in S has degree at least 3. Then $|N[S]| \geq 4$, $\text{ppt}(G) \leq n - 3$, and $\text{ppt}(G) + \text{ppt}(\overline{G}) \leq n - 1$. If $\Delta(G) = 2$, then G is a path. By Proposition 4, $\text{ppt}(P_n) = \lfloor \frac{n}{2} \rfloor$, so $\text{ppt}(P_n) \leq n - 3$ for all $n \geq 6$. For P_3, P_4, P_5 , we have by inspection that $\text{ppt}(P_3) + \text{ppt}(\overline{P_3}) = 2$, $\text{ppt}(P_4) + \text{ppt}(\overline{P_4}) = 4$, and $\text{ppt}(P_5) + \text{ppt}(\overline{P_5}) = 4$. Thus, $\text{ppt}(G) + \text{ppt}(\overline{G}) \leq n - 1$ for all graphs $G \neq P_4$ containing a leaf. The bound is tight for $G = \text{sp}(1, 1, t)$ ($t \geq 2$) since $\text{ppt}(G) = |G| - 3$ by Proposition 13 and $\text{ppt}(\overline{G}) = 2$. \square

Lemma 21. [4] *Let G be a graph such that $\Delta(G) \geq 3$. Then there exists a minimum power dominating set S such that each $s \in S$ has at least two neighbors which are not neighbors of any vertex in $N[S \setminus \{v\}]$.*

The next proposition shows that if $\Delta(G) \geq 3$, $\Delta(\overline{G}) \geq 3$, and $\gamma_P(G) + \gamma_P(\overline{G}) \geq 4$, then $\text{ppt}(G) + \text{ppt}(\overline{G}) \leq n$.

Proposition 22. *Let G be a graph on n vertices such that $\Delta(G) \geq 3$ and $\Delta(\overline{G}) \geq 3$. Then $\text{ppt}(G) + \text{ppt}(\overline{G}) \leq n - (\gamma_P(G) + \gamma_P(\overline{G})) + 4$.*

Proof. By Lemma 21 and the assumption that $\Delta(G) \geq 3$, there is a minimum power dominating set S of G such that each $s \in S$ has at least one neighbor not in $N[S \setminus \{s\}]$. We first show that $Z(G) \leq |N[S]| - \gamma_P(G)$. Recall that $N[S]$ is a zero forcing set of G . For each $s \in S$, choose a $v_s \in N(s)$ such that $v_s \notin N[S \setminus \{s\}]$. Then $N[S] \setminus \{v_1, v_2, \dots, v_{|S|}\}$ is also a zero forcing set since s will force v_s in step one. So, $Z(G) \leq |N[S]| - \gamma_P(G)$.

Similarly, let S' be a minimum power dominating set of G such that each $s' \in S'$ has at least one neighbor not in $N[S' \setminus \{s'\}]$. By the same argument, we have $Z(\overline{G}) \leq |N[S']| - \gamma_P(\overline{G})$. Using the bounds $\text{ppt}(G, S) - 1 \leq n - |N[S]|$ and $\text{ppt}(\overline{G}, S') - 1 \leq n - |N[S']|$ (from inequality (2)), and $n - 2 \leq Z(G) + Z(\overline{G})$ from [6], it follows that

$$\begin{aligned} \text{ppt}(G) + \text{ppt}(\overline{G}) &\leq \text{ppt}(G, S) + \text{ppt}(\overline{G}, S') \\ &\leq 2n + 2 - (|N[S]| + |N[S']|) \\ &\leq 2n + 2 - (Z(G) + Z(\overline{G})) - (\gamma_P(G) + \gamma_P(\overline{G})) \\ &\leq 2n + 2 - (n - 2) - (\gamma_P(G) + \gamma_P(\overline{G})) \\ &= n - (\gamma_P(G) + \gamma_P(\overline{G})) + 4. \quad \square \end{aligned}$$

6 Effects of graph operations on standard power propagation time

Let G_e be a graph obtained from $G = (V, E)$ by subdividing the edge $e \in E$ and let G/e denote the graph resulting from G by contracting the edge e . It is shown in both [2] and [5] that $\gamma_P(G) - 1 \leq \gamma_P(G/e) \leq \gamma_P(G) + 1$ and in [2] that $\gamma_P(G) \leq \gamma_P(G_e) \leq \gamma_P(G) + 1$. We show that the power propagation time may increase or decrease by any amount when subdividing or contracting an edge.

6.1 Edge subdivision

Proposition 23. *For any $t \geq 0$, there exists a graph $G = (V, E)$ and edge $e \in E$ such that $\text{ppt}(G_e) \leq \text{ppt}(G) - t$.*

Proof. Construct the graph G in the following way: Starting with the path $P_\ell = (v_1, v_2, \dots, v_\ell)$, ($\ell \geq 7$), add two leaves to vertex v_1 and add two leaves to

vertex v_ℓ so that vertices v_1 and v_ℓ become strong support vertices. Add one leaf to vertex $v_{\ell-1}$ and add one leaf to vertex $v_{\ell-2}$. (See Figure 3.) Then $\{v_1, v_\ell\}$ is the unique power dominating set of G and $\text{ppt}(G) = \ell - 2$. For $e = \{v_{\ell-2}v_{\ell-1}\}$, we consider the graph G_e . Note that $\gamma_p(G_e) = 3$ because $v_1, v_\ell \in S$ for any power dominating set S and $\{v_1, v_\ell\}$ is not a power dominating set. For $S = \{v_1, v_{\ell-2}, v_\ell\}$, $\text{ppt}(G_e, S) = \lceil \frac{\ell-4}{2} \rceil$. By choosing $\ell \geq 2t+1$, $\text{ppt}(G_e) \leq \text{ppt}(G) - t$. \square

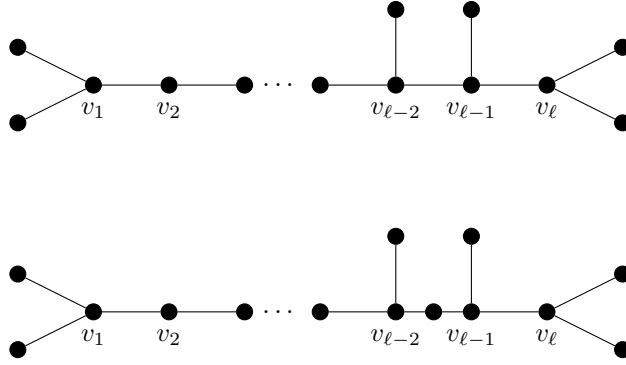


Figure 3: Graphs G and G_e in Proposition 23.

Similarly, subdividing an edge can cause the power propagation time to increase by any amount, as demonstrated by the following proposition.

Proposition 24. *For any $t \geq 0$, there exists a graph $G = (V, E)$ and edge $e \in E$ such that $\text{ppt}(G_e) \geq \text{ppt}(G) + t$.*

Proof. Let G be a graph on $n \geq 8$ vertices constructed from the cycle $(v_1, v_2, \dots, v_{n-4})$ by adding the edges $\{v_1, v_{n-3}\}$, $\{v_1, v_{n-2}\}$, $\{v_1, v_{n-1}\}$, $\{v_2, v_{n-1}\}$, and $\{v_n, v_{n-1}\}$. Let $e = \{v_2, v_{n-1}\}$, and consider G_e . The set $\{v_1\}$ is the unique minimum power dominating set of G and $\text{ppt}(G) = \lfloor \frac{n-4}{2} \rfloor$. The set $\{v_1\}$ is also the unique minimum power dominating set of G_e and $\text{ppt}(G_e) = n - 4$. So, by choosing $n \geq 2t + 4$, $\text{ppt}(G_e) \geq \text{ppt}(G) + t$. \square

6.2 Edge Contraction

Proposition 25. *For any $t \geq 0$, there exists a graph $H = (V, E)$ and edge $e \in E$ such that $\text{ppt}(H/e) \geq \text{ppt}(H) + t$.*

Proof. From Proposition 23, there exist graphs G and G_e such that $\text{ppt}(G_e) \leq \text{ppt}(G) - t$. Let $H = G_e$ and $H/e = G$. Then $\text{ppt}(H/e) \geq \text{ppt}(H) + t$. \square

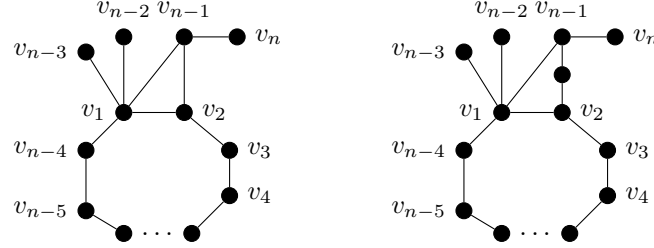


Figure 4: Graphs G and G_e in Proposition 24.

Proposition 26. *For any $t \geq 0$, there exists a graph $H = (V, E)$ and edge $e \in E$ such that $\text{ppt}(H/e) \leq \text{ppt}(H) - t$.*

Proof. From Proposition 24, there exist graphs G and G_e such that $\text{ppt}(G_e) \geq \text{ppt}(G) + t$. Let $H = G$ and $H/e = G_e$. Then $\text{ppt}(H/e) \leq \text{ppt}(H) - t$. \square

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